Question 1.

Let $M_n(\mathbf{C})$ be the algebra of all $n \times n$ matrices of complex entries. Suppose $E, F \in M_n(\mathbf{C})$ are projections, i.e., E, F are selfadjoint matrices so that $E = E^2$ and $F = F^2$. Assume that $||E - F|| \leq 1/2$, where $|| \cdot ||$ is the operator norm. Show that there is a unitary matrix U such that $UEU^* = F$.

Question 2. Let E be a measurable subset of R^d with $mes(E) < +\infty$, g is a measureable function on E.

(a) Suppse that for all $f \in L^1(E)$, there holds $f(x)g(x) \in L^1(E)$. Then $g \in L^\infty(E)$.

(b) Given $g \in L^\infty(E)$, we have

$$\|g\|_{L^\infty} = \sup_{\|f\|_{L^1}=1} \{|\int_E f(x)g(x)dx|\}$$

Question 3.

Let u be a harmonic function over $\{z \in \mathbf{C} \mid 0 < |z| < 1\}$, prove there exist constants α, β such that for all r,

$$\frac{1}{2\pi}\int_{|z|=r} u d\theta = \alpha \log r + \beta.$$

Question 4.

Let $D = (0,1) \times (0, \infty)$. Find all the solutions:

$$\begin{cases} \Delta u &= 0 \text{ in } D \\ u &> 0 \text{ in } D \\ u &= 0 \text{ on } \partial D \end{cases}$$

Question 5. Consider Legendre polynomials (up to a multiple constant), i.e.,

$$P_0(x) = 1,$$
$$P_n(x) = \frac{d^n}{dx^n}\left((x^2-1)^n\right), \quad n \geq 1,$$

1) Show that P_n is orthogonal to all polynomials with degree less than n in the sense

$$\int_{-1}^{1} P_n(x)P(x)dx = 0 \quad \text{if } \deg P < n.$$

2) Show that all roots of Legendre polynomial $P_n(x)$ are real, simple and lie in $(-1,1)$.

3) Consider approximation of integral $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$. How to choose n points $x_1, x_2, \cdots, x_n$ on $(-1,1)$ and the weights $w_1, w_2, \cdots, w_n$ such that the approximation of the above integral is exact for all polynomials with degree $\leq 2n-1$?

4) Given a function, say $f(x) = \sin x, e^x$ or e^{-x^2}, how to find a degree 2 polynomial $P(x)$ such that $\int_{-1}^{1} |f(x) - P(x)|^2 dx$ is minimized?